Skin Friction Law for Compressible Turbulent Flow

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An algebraic skin-friction law is derived for adiabatic, compressible, equilibrium, turbulent boundary-layer flow. An outer solution in terms of the Clauser defect stream function is joined to an inner empirical expression composed of compressible laws of the wall and wake. The modified Crocco temperature-velocity relationship and the Clauser eddy viscosity model are used in the outer solution. The skin-friction law pertains for all pressure gradients in the incompressible through supersonic range and for small pressure gradients in the hypersonic range. Excellent comparisons with experiment are obtained in the appropriate parameter ranges.

Introduction

OVER 30 years ago Clauser[1] observed that the experimental tangential velocity distributions for incompressible turbulent boundary layers on flat plates are similar when plotted in terms of the deficit in velocity from the boundary-layer edge value and normalized with the wall shear stress velocity. This observed similarity pertains to the outer part of the boundary layer and is independent of the Reynolds number. As a result, the normalized velocity defect profiles of a given incompressible turbulent flat-plate boundary layer do not change with distance along the plate and are said to be self-similar.

Clauser reasoned that the velocity defect profiles of turbulent boundary layers with streamwise pressure gradients should also be self-similar if the pressure and skin-friction forces were, in his terminology, in equilibrium. He showed that equilibrium occurs when the pressure gradient normalized with the displacement thickness varies in direct proportion to the wall shear stress and demonstrated experimentally the existence of two incompressible equilibrium boundary layers with adverse pressure gradients. He also identified a simple, highly accurate eddy viscosity model for the outer part of the equilibrium boundary layer.

Ten years later Mellor and Gibson[2] successfully solved the exact incompressible equilibrium boundary-layer problem numerically in terms of Clauser's defect formulation. They used the Clauser eddy viscosity model in the outer part of the boundary layer and the Prandtl model in the inner part. In addition to the exact solution, Mellor and Gibson obtained an extremely accurate approximate solution in the limit of vanishing shear stress velocity to edge velocity ratio with a governing equation that is linear except where the Prandtl eddy viscosity is used. The method is difficult to apply in the inner part of the boundary layer, where the defect velocity becomes large and the nonlinear effects become increasingly important.

Barnwell et al.[3] showed that robust solutions to both the exact and approximate incompressible equilibrium boundary-layer problems can be obtained by joining numerical defect velocity solutions for the outer part of the boundary layer to empirical expressions involving the laws of the wall and wake[4] for the inner part. Two advantages of the new approach are that the inner part of the boundary layer, where gradients are high, does not have to be resolved computationally and that the use of the inner-layer eddy viscosity model, which generally cannot be validated by direct experimental observation, is avoided.

Wahls et al.[5] and Wahls[6] have formulated the equilibrium and nonequilibrium problems for adiabatic compressible turbulent boundary-layer flow and have obtained numerical solutions to the equilibrium problem. The energy equation is replaced with the modified Crocco temperature-velocity relationship, and a modified form of the van Driest law of the wall for compressible flow[7] is used. It is shown that an extremely accurate approximate solution exists for compressible flow and that this solution has a linear governing equation similar to that for incompressible flow. A major result of Ref. 5 is the determination of an expression for the law of the wake for compressible flow.

In the present paper a closed-form solution is derived for the approximate formulation of Refs. 5 and 6 for adiabatic, compressible, equilibrium flows with arbitrary pressure gradients and edge Mach numbers that are supersonic or less, or with small pressure gradients and hypersonic edge Mach numbers. The skin-friction law is obtained from this solution.

There are several other methods that use a law of the wall formulation near the wall. Wall function methods such as that of Viegas et al.[8] simply patch the numerical and analytical solutions at the first grid point and employ an eddy viscosity model throughout the boundary layer. The method of Walker et al.[9] formally matches the outer limit of the inner solution to the inner limit of the outer solution. The inner solution of Ref. 9 does not include the law of the wake, and the matching is performed in the law of the wall region much closer to the wall than the present match point so that an inner as well as an outer eddy viscosity model must be specified.

Analysis

In this treatment the zero-order form of the equation for equilibrium compressible boundary-layer flow derived in Ref. 5 is integrated in closed form. The present solution pertains for all pressure gradients in the speed range from incompressible to supersonic and for small pressure gradients in the hypersonic range.

Presented as Paper 89-1864 at the AIAA 20th Fluid Dynamics, Plasma Dynamics and Lasers Conference, June 12–14, 1989; received June 29, 1989; revision received Feb. 21, 1990.

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Basic Equations

The continuity and tangential momentum equations for compressible, turbulent boundary-layer flow are

$$\frac{\partial(\rho U)}{\partial x}+\frac{\partial(\rho v)}{\partial y}=0$$

$$\rho U\frac{\partial U}{\partial x}+\rho v\frac{\partial U}{\partial y}-\rho_e U_e\frac{\mathrm{d}U_e}{\mathrm{d}x}=\frac{\partial}{\partial y}\left[(\mu+\mu_t)\frac{\partial U}{\partial y}\right]$$

where x and y are the tangential and normal coordinates, respectively, U and v are the respective velocity components, ρ is the density, μ and μ_t are the viscosity and turbulent eddy viscosity, respectively, and the subscript e refers to the boundary-layer edge. In the defect part of the boundary layer μ is neglected and μ_t is modeled as

$$\mu_t=K(x,y)\rho U_e\delta_i^*$$

where K is a general nondimensional function of x and y, and δ_i^* is the incompressible displacement thickness.

The modified Crocco equation for adiabatic walls,

$$\frac{T_w}{T}=\frac{\rho}{\rho_w}=1+\frac{r}{2}(\gamma-1)(U/c)^2$$

is used as the energy equation, where w designates properties at the wall, γ is the ratio of specific heats, c is the speed of sound, and $r=Pr^{1/3}$ is the recovery factor, where Pr is the Prandtl number.

Defect Stream Function Formulation

The defect stream function formulation of Clauser[1] is modified to account for compressibility. The defect stream function $f[x,\eta(x,y)]$ is defined as

$$\frac{\partial f}{\partial \eta}=\frac{U-U_e}{U^*}$$

where

$$\eta=\frac{1}{\Delta}\int^y\frac{\rho}{\rho_e}\,\mathrm{d}y$$

The shear-stress velocity U^* is

$$U^*=\sqrt{\tau_w/\rho_w}$$

where ρ_w and τ_w are the density and shear stress at the wall, respectively. The boundary-layer thickness parameter Δ is

$$\Delta=-\int_0^\infty\frac{\rho}{\rho_e}\frac{\partial f}{\partial \eta}\,\mathrm{d}y=\frac{U_e}{U^*}\int_0^\infty\frac{\rho}{\rho_e}\left(1-\frac{U}{U_e}\right)\mathrm{d}y=\frac{U_e\delta_v^*}{U^*}$$

or

$$\Delta U^*=U_e\delta_v^*$$

where the density-weighted velocity thickness δ_v^* is

$$\delta_v^*=\int_0^\infty\frac{\rho}{\rho_e}\left(1-\frac{U}{U_e}\right)\mathrm{d}y$$

The density ratio ρ_e/ρ is expressed in terms of the defect stream function f as

$$\frac{\rho_e}{\rho}=1-\left(2+\frac{U^*}{U_e}\frac{\partial f}{\partial \eta}\right)\frac{U^*}{U_e}\left(\frac{\rho_e}{\rho_w}-1\right)\frac{\partial f}{\partial \eta}$$

The governing equation for f is obtained from the tangential momentum equation. An accurate approximate form with a first integral that is valid for all speeds is obtained for the conditions

$$\frac{U^*}{U_e}\ll 1,\qquad \frac{U^*}{U_e}M_e\le 0(1)$$

where M_e is the edge Mach number. The derivatives of U are

$$\frac{\partial U}{\partial x}=\frac{\mathrm{d}U_e}{\mathrm{d}x}\left(1+\frac{U^*}{U_e}\frac{\partial f}{\partial \eta}\right)+U_e\frac{\partial}{\partial x}\left(\frac{U^*}{U_e}\frac{\partial f}{\partial \eta}\right)+U^*\frac{\partial^2 f}{\partial \eta^2}\frac{\partial \eta}{\partial x}$$

$$\times\frac{\partial U}{\partial y}=\frac{\rho U^*}{\rho_e\Delta}\frac{\partial^2 f}{\partial \eta^2}$$

and the flux ρv is

$$\rho v=-\frac{\mathrm{d}}{\mathrm{d}x}(\rho_e U_e\Delta)\left(\eta+\frac{U^*}{U_e}f\right)-\rho_e U_e\Delta\frac{\partial}{\partial x}\left(\frac{U^*}{U_e}f\right)$$
$$-\rho_e U_e\Delta\left(1+\frac{U^*}{U_e}\frac{\partial f}{\partial \eta}\right)\frac{\partial \eta}{\partial x}$$

It will be shown that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{U^*}{U_e}\right)\sim\left(\frac{U^*}{U_e}\right)^2$$

and hence can be neglected. To first order in U^*/U_e the tangential momentum equation is

$$\frac{1}{\beta}\frac{\rho_e}{\rho_w}\frac{\partial^2 f}{\partial s\,\partial \eta}=\frac{1}{\beta}\frac{\rho_e}{\rho_w}\omega\frac{\partial}{\partial \eta}\left(\frac{\rho^2}{\rho_e^2}K\frac{\partial^2 f}{\partial \eta^2}\right)$$
$$-\left(1-M_e^2+\frac{U_e\dot{\Delta}}{\dot{U}_e\Delta}\right)\eta\frac{\partial^2 f}{\partial \eta^2}+2\frac{\rho_e}{\rho_w}\frac{\partial f}{\partial \eta}$$

or

$$\frac{\partial}{\partial \eta}\left\{\frac{1}{\beta}\frac{\rho_e}{\rho_w}\omega\frac{\rho^2}{\rho_e^2}K\frac{\partial^2 f}{\partial \eta^2}-\left(1-M_e^2+\frac{U_e\dot{\Delta}}{\dot{U}_e\Delta}\right)\left(\eta\frac{\partial f}{\partial \eta}-f\right)\right.$$
$$\left.+2\frac{\rho_e}{\rho_w}f-\frac{1}{\beta}\frac{\rho_e}{\rho_w}\frac{\partial f}{\partial s}\right\}=0 \tag{1}$$

where

$$s=\int_0^x\frac{U^*}{U_e\Delta}\,\mathrm{d}x$$

and

$$\omega=\frac{\delta_i^*}{\delta_v^*}$$

The dot represents differentiation with respect to x, and the pressure gradient parameter β is defined as

$$\beta=\frac{\delta_v^*}{\tau_w}\frac{\mathrm{d}p}{\mathrm{d}x}$$

where p is the pressure.

There are two surface boundary conditions and one far-field boundary condition. The shear stress boundary condition is

$$\tau=\tau_w$$

or

$$\lim_{\eta\to 0}K\left(\frac{\rho}{\rho_e}\right)^2\frac{\partial^2 f}{\partial \eta^2}=\frac{\rho_w}{\rho_e}\frac{1}{\omega}$$

From the equation for the flux ρv it is concluded that f

vanishes at the surface:

$$f(s,0) = 0$$

The far-field condition is obtained from the definition of the boundary-layer thickness parameter Δ:

$$\Delta = -\int_0^\infty \frac{\rho}{\rho_e}\frac{\partial f}{\partial \eta}\,dy = -\Delta\int_0^\infty \frac{\partial f}{\partial \eta}\,d\eta = \Delta[f(s,0) - f_\infty(s)]$$

It follows that

$$f_\infty(s) = -1$$

With the three boundary conditions the governing Eq. (1) is integrated across the boundary layer to obtain

$$1 - M_e^2 + \frac{U_e\,\dot{\Delta}}{\dot{U}_e\,\Delta} = -2\frac{\rho_e}{\rho_w} - \frac{1}{\beta}$$

Therefore, the first integral of Eq. (1) for arbitrary η is

$$\frac{\rho_e}{\rho_w}\frac{\partial f}{\partial s} = \frac{\rho_e}{\rho_w}\omega\left(\frac{\rho}{\rho_e}\right)^2 K\frac{\partial^2 f}{\partial \eta^2} + \left(1 + 2\frac{\rho_e}{\rho_w}\beta\right)\eta\frac{\partial f}{\partial \eta} - f - 1$$

The strong dependence of the coefficients on the ratio ρ_e/ρ_w can be removed with the transformation

$$\hat{s} = \int^s \frac{\rho_w}{\rho_e}\,ds, \qquad \hat{\eta} = \left[\frac{\rho_w}{\rho_e}\right]^{1/2}\eta$$

The resulting equation is

$$\frac{\partial f}{\partial \hat{s}} = \omega\left(\frac{\rho}{\rho_e}\right)^2 K\frac{\partial^2 f}{\partial \hat{\eta}^2} + (1 + 2\hat{\beta})\hat{\eta}\frac{\partial f}{\partial \hat{\eta}} - f - 1 \tag{2}$$

where

$$\hat{\beta} = \frac{\rho_e}{\rho_w}\beta\left\{1 - \frac{1}{2r}\left[1 - \frac{\rho_w}{\rho_e}\right]\left[1 - \frac{\rho_w}{\rho_e}(1 - r)\right]\right\}$$

To lowest order in U^*/U_e, the ratios ρ/ρ_e and ω are

$$\frac{\rho}{\rho_e} = \left(1 - \varepsilon\frac{\partial f}{\partial \hat{\eta}}\right)^{-1}$$

$$\omega = 1 + \varepsilon\int_0^\infty \left(\frac{\partial f}{\partial \hat{\eta}}\right)^2 d\hat{\eta}$$

where

$$\varepsilon = 2\frac{U^*}{U_e}\left[\frac{\rho_e}{\rho_w}\right]^{1/2}\left(1 - \frac{\rho_w}{\rho_e}\right)$$

For edge Mach numbers from incompressible to supersonic this parameter is small so that the ratios above are approximately one, and Eq. (2) is linear. However, because the parameter ε has the limiting form

$$\lim_{M_e\to\infty} \varepsilon = 2\left[\frac{r(\gamma - 1)}{2}\right]^{1/2}\frac{U^*}{U_e}M_e$$

it is not small for large values of the edge Mach number; hence, Eq. (2) is nonlinear.

It is shown in Ref. 5 that numerical solutions to the nonlinear, equilibrium form of Eq. (2) and the equilibrium form of the exact governing equation are virtually the same. These results indicate that Eq. (2), which is much simpler than the exact equation, is an effective governing equation for the flow.

Coles[10] uses an ε-like parameter to correlate experimental data for compressible boundary layers. His parameter, which corresponds to ε^2, is $M_\infty^2 C_f$, where M_∞ and C_f are the freestream Mach number and skin-friction coefficient, respectively.

An additional transformation is necessary to obtain a governing equation that is linear in the hypersonic range. First note that Eq. (2) can be written as

$$\frac{\partial f}{\partial \hat{s}} = \omega K\frac{\partial}{\partial \hat{\eta}}\left(\frac{\rho}{\rho_e}\frac{\partial f}{\partial \hat{\eta}}\right) + (1 + 2\hat{\beta})\hat{\eta}\frac{\partial f}{\partial \hat{\eta}} - f - 1$$

The transformation is

$$\tilde{s} = \hat{s}, \qquad \tilde{\eta} = \int \frac{\rho_e}{\rho}\,d\hat{\eta} = \hat{\eta} - \varepsilon f$$

In these coordinates the ratios ρ/ρ_e and ω are

$$\frac{\rho}{\rho_e} = 1 + \varepsilon\frac{\partial f}{\partial \tilde{\eta}}$$

$$\omega = 1 + \varepsilon\int_0^\infty \left(1 + \varepsilon\frac{\partial f}{\partial \tilde{\eta}}\right)^{-1}\left(\frac{\partial f}{\partial \tilde{\eta}}\right)^2 d\tilde{\eta}$$

The governing equation is

$$\frac{\partial f}{\partial \tilde{s}} = \omega K\frac{\partial^2 f}{\partial \tilde{\eta}^2} + (1 + 2\tilde{\beta})(\tilde{\eta} - \tilde{\eta}_0)\frac{\partial f}{\partial \tilde{\eta}} - f - 1 - \varepsilon\tilde{\beta}f\frac{\partial f}{\partial \tilde{\eta}} \tag{3}$$

where

$$\tilde{\beta} = 2\beta\frac{(1 - r)}{r}\left(\frac{\rho_e}{\rho_w} - 1\right)$$

$$\tilde{\eta}_0 = \varepsilon/(1 + 2\tilde{\beta})$$

This equation is linear if the inequality

$$\varepsilon|\tilde{\beta}| \ll 1 \tag{4}$$

is satisfied. This inequality is satisfied for all pressure gradients in the incompressible to supersonic range and for small pressure gradients in the hypersonic range so that Eq. (2) is linear for most cases of practical interest. Note that this equation is linear for all cases with a Prandtl number of one.

Equilibrium Governing Equation

As discussed earlier, Clauser defined a turbulent equilibrium boundary layer as one for which an equilibrium exists between the pressure and the skin-friction forces so that the normalized defect velocity profiles at each streamwise station are self-similar. For such flows the defect stream function f has no streamwise variation so that the left side of Eq. (3) vanishes. This can occur only if the coefficients on the right side of Eq. (3) do not depend on the streamwise location. Therefore, equilibrium flow is possible if inequality (4) is satisfied, the coefficient K is independent of $\tilde{s}$, and the pressure gradient parameter $\tilde{\beta}$ is constant. Note that $\tilde{\eta}_0$ and ω are effectively constant if inequality (4) is satisfied. In this paper it is assumed that the coefficient K is equal to the Clauser constant k. The equilibrium governing equation is written as

$$\omega k\frac{d^2 f}{d\tilde{\eta}^2} + (1 + 2\tilde{\beta})(\tilde{\eta} - \tilde{\eta}_0)\frac{df}{d\tilde{\eta}} - f - 1 = 0$$

The equilibrium relationship between the pressure and skin-friction forces is determined by the value of $\tilde{\beta}$.

Outer Solution

As Mellor and Gibson[2] discovered for the incompressible problem, the governing equation for compressible, equilibrium flow is a confluent hypergeometric equation. The independent variable of Mellor and Gibson for the present problem is

$$N = \frac{1+2\hat{\beta}}{\omega k}\frac{(\tilde{\eta}-\tilde{\eta}_0)^2}{2}$$

Define the dependent variable

$$\hat{f} = (1+f)e^N$$

The governing equation in terms of these variables is the confluent hypergeometric equation

$$N\frac{d^2\hat{f}}{dN^2} + (b-N)\frac{d\hat{f}}{dN} - a\hat{f} = 0$$

with the parameters

$$b = \frac{1}{2}, \qquad a = \frac{1}{2}\left(1+\frac{1}{1+2\hat{\beta}}\right)$$

The solution to the confluent hypergeometric equation is[11]

$$\hat{f} = CM(a,b,N) + DN^{1-b}M(1+a-b,2-b,N)$$

where C and D are arbitrary constants, and $M(a,b,N)$ is Kummer's function:

$$M(a,b,N) = 1 + \frac{a}{b}N + \frac{a(a+1)}{b(b+1)}\frac{N^2}{2!} + \cdots$$

The coefficients C and D can be related with the far-field boundary condition. For large values of N, Kummer's function varies as

$$M(a,b,N) = \frac{\Gamma(b)}{\Gamma(a)}e^N N^{a-b}[1+0(N^{-1})]$$

It follows that

$$\lim_{N\to\infty}\hat{f} = \left\{C\frac{\Gamma(b)}{\Gamma(a)} + D\frac{\Gamma(2-b)}{\Gamma(1-a+b)}\right\}e^N N^{a-b} = 0$$

so that

$$D = -C\frac{a}{b}\frac{\Gamma(1+a-b)}{\Gamma(1+a)}$$

The coefficient C can be evaluated with the surface boundary condition $f(0)=0$. The defect stream function is

$$f = -1 + C\{M(a,\tfrac{1}{2},N) - \frac{a}{[(a-\frac{1}{2})\omega k]^{1/2}}\frac{\Gamma(\frac{1}{2}+a)}{\Gamma(1+a)}(\tilde{\eta}-\tilde{\eta}_0)M(\tfrac{1}{2}+a,\tfrac{3}{2},N)\}e^{-N}$$

where

$$C = \left\{M(a,\tfrac{1}{2},N_0) + \frac{a}{[(a-\frac{1}{2})\omega k]^{1/2}}\frac{\Gamma(\frac{1}{2}+a)}{\Gamma(1+a)}\tilde{\eta}_0 \times M(\tfrac{1}{2}+a,\tfrac{3}{2},N_0)\right\}^{-1}e^{N_0}$$

$$N_0 = \frac{1+2\hat{\beta}}{\omega k}\frac{\tilde{\eta}_0^2}{2}$$

The first and second derivatives of f are

$$\frac{df}{d\tilde{\eta}} = -C\left\{\frac{a}{[(a-\frac{1}{2})\omega k]^{1/2}}\frac{\Gamma(\frac{1}{2}+a)}{\Gamma(1+a)} \times M(a-\tfrac{1}{2},\tfrac{1}{2},N) - \frac{(\tilde{\eta}-\tilde{\eta}_0)}{\omega k}M(a,\tfrac{3}{2},N)\right\}e^{-N}$$

and

$$\frac{d^2f}{d\tilde{\eta}^2} = C\left\{\frac{1}{\omega k}M(a-1,\tfrac{1}{2},N) + \frac{a(1-a)}{[(a-\frac{1}{2})\omega k]^{3/2}}\frac{\Gamma(\frac{1}{2}+a)}{\Gamma(1+a)}(\tilde{\eta}-\tilde{\eta}_0)M(a-\tfrac{1}{2},\tfrac{3}{2},N)\right\}e^{-N}$$

The first derivative gives the velocity profile for turbulent compressible equilibrium flow in the outer region. It is noted that a slightly different method for enforcing the inner boundary condition is used to obtain the numerical solutions presented in Refs. 3 and 5.

The gamma functions $\Gamma(\frac{1}{2}+a)$ and $\Gamma(1+a)$ are used because they both have values very near 1 for all unfavorable pressure gradients and because their ratio can be estimated easily for favorable pressure gradients. For unfavorable pressure gradients the parameter a satisfies the inequality

$$1 \geq a \geq \tfrac{1}{2}, \qquad 0 \leq \hat{\beta} \leq \infty$$

so that

$$\tfrac{3}{2} \geq \tfrac{1}{2}+a \geq 1, \quad 2 \geq 1+a \geq \tfrac{3}{2}; \quad 0 \leq \hat{\beta} \leq \infty$$

Gamma functions have values of 1 for arguments of 1 and 2 and values between 0.8856 and 1 for arguments between 1 and 2. For large favorable pressure gradients the ratio of the gamma functions is

$$\lim_{\hat{\beta}\to-1/2}\frac{\Gamma(\frac{1}{2}+a)}{\Gamma(1+a)} = \sqrt{2(1+2\hat{\beta})}\left\{1-\frac{3}{4}[1+2\hat{\beta}] + 0([1+2\hat{\beta}]^2)\right\}$$

Inner Empirical Formulation

The tangential velocity near the wall is obtained from the combined law of the wall and wake, which is written as

$$U = U^*\{g(y^+) + h(\tilde{\eta})\}$$

where

$$\tilde{\eta} = \frac{y}{\Delta}, \qquad y^+ = \frac{\rho_w U^* y}{\mu_w} = \frac{Re_\delta^*}{\omega}\tilde{\eta}, \qquad Re_\delta^* = \frac{\rho_w U_e \delta_i^*}{\mu_w}$$

Law of the Wall

The law of the wall for the adiabatic flow of a calorically perfect compressible gas is

$$g(y^+) = \left[\frac{2}{r(\gamma-1)}\right]^{1/2}\frac{c_w}{U^*}\sin\left\{\left[\frac{r(\gamma-1)}{2}\right]^{1/2} \times \frac{U^*}{c_w}\left(\frac{1}{K}\ell n\, y^+ + B\right)\right\}$$

where c_w is the wall speed of sound. The derivation of this equation is similar to that of van Driest[7] for a Prandtl number of one. This form pertains in the outer part of the wall layer where the inner empirical expression is joined to the outer solution. It does not pertain in the laminar sublayer, where a more detailed form of the function $g(y^+)$ is needed.

Law of the Wake

In this treatment it is required only that the law of the wake be defined in the inner part of the boundary layer. Consequently, the form of the law of the wake used here contains only a leading term:

$$h(\tilde{\eta}) = \frac{6}{\kappa}W\tilde{\eta}^2$$

An empirical expression for incompressible equilibrium turbulent flow has been established by White[12] as

$$W_i(\beta) = \Pi\left(\frac{\Delta}{\delta}\right)^2, \quad \frac{\Delta}{\delta} = \frac{1+\Pi}{\kappa}, \quad \Pi = \frac{4}{5}\left(\frac{1}{2}+\beta\right)^{\frac{3}{4}}$$

It is demonstrated numerically in Ref. 5 that, for compressible flow,

$$W(\beta) = W_i(\beta)\left(\frac{\rho_w}{\rho_e}\right)^\alpha, \quad \alpha = 2 - e^{-5/4(1+2\beta)^{1/2}}$$

Skin-Friction Law

The inner empirical formulation and outer solution are joined by equating the two expressions for $\partial U/\partial y$ and U and determining the match point location $\tilde{\eta}_m$ and the value of the ratio U^*/U_e. This is the same procedure that Clauser[1] used to obtain his skin-friction law for incompressible, flat-plate flow. The derivative equation is

$$\frac{1}{U^*}\frac{\partial U}{\partial y}(y_m) = \left[\frac{\partial\tilde{\eta}}{\partial y}\frac{\partial}{\partial\tilde{\eta}}\{g(y^+) + h(\tilde{\eta})\}\right]_m = \left[\frac{\partial\tilde{\eta}}{\partial y}\frac{\partial}{\partial\tilde{\eta}}\left(\frac{\partial\tilde{\eta}}{\partial\eta}\frac{\mathrm{d}f}{\mathrm{d}\tilde{\eta}}\right)\right]_m$$

where the subscript m designates the match point, and where

$$\tilde{\eta}_m = \left[\frac{\rho_w}{\rho_e}\right]^{1/2}\bar{\eta}_m$$

Therefore, the deritive equation, which determines the match point, can be written as

$$\left[\frac{\rho_e}{\rho_w}\right]^{1/2}\cos\left\{\left[\frac{r(\gamma-1)}{2}\right]^{1/2}\frac{U^*}{c_w}\left(\frac{1}{\kappa}\ell n y_m^+ + B\right)\right\}\frac{1}{\kappa\tilde{\eta}_m} + \frac{12}{\kappa}W\left[\frac{\rho_e}{\rho_w}\right]^{3/2}\tilde{\eta}_m = \left(\frac{\rho_e}{\rho_m}\right)^2\frac{\mathrm{d}^2 f}{\mathrm{d}\tilde{\eta}^2}(\tilde{\eta}_m) \tag{5}$$

The equation for joining the two velocity expressions is

$$U = U_e + U^*\frac{\mathrm{d}f}{\mathrm{d}\eta}(\eta_m) = U^*\{g(y_m^+) + h(\tilde{\eta}_m)\}$$

which can be written as

$$\frac{U^*}{U_e} = \left\{g(y_m^+) + h(\tilde{\eta}_m) - \frac{\rho_e}{\rho_m}\left[\frac{\rho_w}{\rho_e}\right]^{1/2}\frac{\mathrm{d}f}{\mathrm{d}\tilde{\eta}}(\tilde{\eta}_m)\right\}^{-1} \tag{6}$$

This equation gives the skin-friction law. Note that

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{U^*}{U_e}\right) = -\left(\frac{U^*}{U_e}\right)^2\left[\dot{g} + \dot{h} - \frac{\mathrm{d}\dot{f}}{\mathrm{d}\eta}\right]_m$$

Results

Results for skin friction obtained with the present analysis are compared with experimental values for equilibrium boundary layers. The values used for the Pr and the Clauser constant k are 0.72 and 0.016, respectively, and the values for the von Kármán constant κ and B are 0.41 and 4.9, respectively. The value used for the ratio of specific heats γ is 1.4.

Comparisons of skin-friction results for incompressible flat-plate flow as well as for flows with both favorable and adverse pressure gradients are presented in Table 1. Data for the six experiments were obtained from Ref. 13, the data volume for the first Stanford conference, and include the two data sets Clauser used to develop the equilibrium boundary-layer concept. The slight differences in the analytical and numerical results are due to the difference in the way the inner boundary condition is enforced.

The present results are compared with experimental skin-friction values for compressible flow over a flat plate in Table 2. Comparisons of results for a favorable and an adverse pressure gradient are also included. The experimental data were obtained from Ref. 14. The numerical results were calculated with the method of Ref. 5. Again, the agreement is excellent.

Comparisons of the analytic and experimental results for two of the velocity profiles are presented in Fig. 1 in outer-layer variables. These results are presented in Fig. 2 in law-of-the-wall coordinates. It is seen that the excellent agreement on skin-friction for compressible flow was achieved even though the law of the wall used in the analytic solution was only a fair approximation of experiment.

The present procedure for joining the outer solution and inner empirical expression is equivalent to finding the value of $\tilde{\eta}_m$ that produces the minimum value of the function for

Table 1 Skin friction for incompressible equilibrium flow

Case	β	Re_δ^*	U^*/U_e experiment (Ref. 13)	U^*/U_e numerical (Ref. 3)	U^*/U_e analytical [Eqs. (5) and (6)]
1312	−0.23	5447.0	0.0411	0.0407	0.0401
1410	0	6754.2	0.0388	0.0386	0.0382
1422	0	19,321.1	0.0350	0.0351	0.0348
2205	1.89	20,230.2	0.0319	0.0301	0.0296
2305	7.53	30,692.5	0.0231	0.0229	0.0225
2402	3.03	42,832.9	0.0267	0.0265	0.0265

Table 2 Skin friction for compressible equilibrium flow

Case	M_e	β	Re_δ^*	U^*/U_e experiment (Ref. 14)	U^*/U_e numerical (Ref. 5)	U^*/U_e analytical [Eqs. (5) and (6)]
5301/0601	2.578	0	5295	0.0426	0.0431	0.0424
5301/1302	4.544	0	902	0.0544	0.0555	0.0533
5501/0101	1.724	0	4719	0.0414	0.0420	0.0414
5801/0101	1.947	−0.16	7949	0.0397	0.0412	0.0407
5801/0501	1.918	0.22	8596	0.0395	0.0391	0.0384

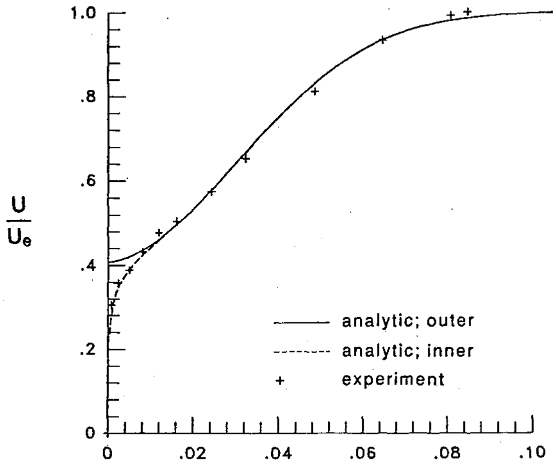

Fig. 1a Tangential velocity profile in outer variables, incompressible flow, $\beta = 7.53$, $Re_\delta^* = 30{,}692.5$.

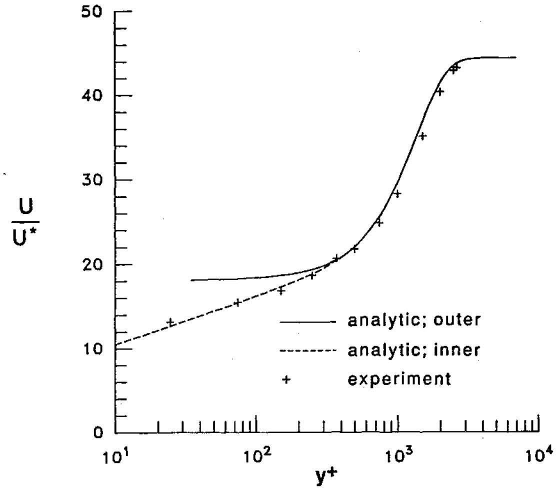

Fig. 2a Tangential velocity profile in law of the wall variables, incompressible flow, $\beta = 7.53$, $Re_\delta^* = 30{,}692.5$.

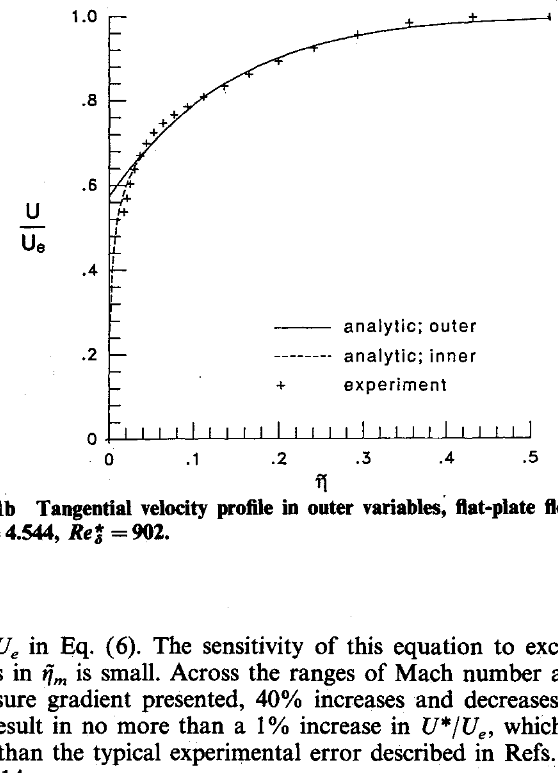

Fig. 1b Tangential velocity profile in outer variables, flat-plate flow, $M_e = 4.544$, $Re_\delta^* = 902$.

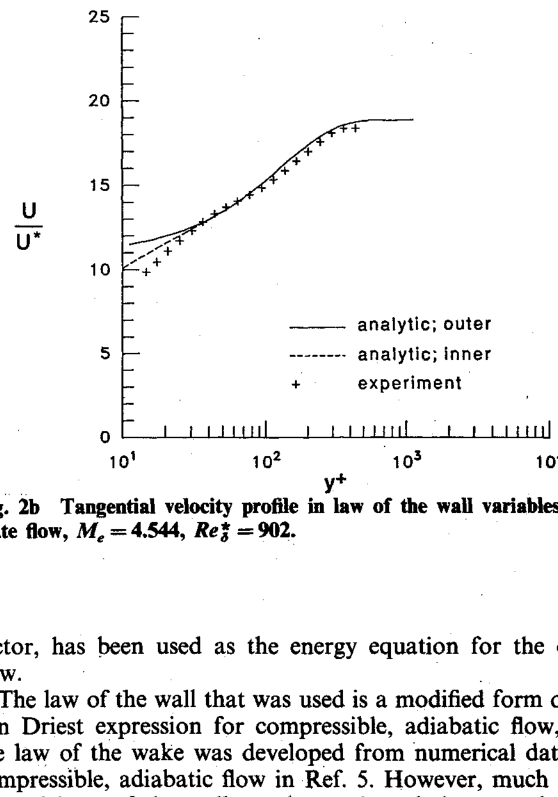

Fig. 2b Tangential velocity profile in law of the wall variables, flat-plate flow, $M_e = 4.544$, $Re_\delta^* = 902$.

U^*/U_e in Eq. (6). The sensitivity of this equation to excursions in $\tilde{\eta}_m$ is small. Across the ranges of Mach number and pressure gradient presented, 40% increases and decreases in $\tilde{\eta}_m$ result in no more than a 1% increase in U^*/U_e, which is less than the typical experimental error described in Refs. 13 and 14.

Conclusions

An algebraic skin-friction law has been derived for adiabatic, compressible, equilibrium turbulent boundary-layer flow. This law results from the joining of an outer equilibrium solution for the tangential velocity with an inner empirical expression composed of the laws of the wall and wake. The need to specify an inner-layer eddy viscosity model, which generally cannot be validated experimentally, is completely avoided.

It has been shown that the governing equation for the outer flow expressed in terms of the Clauser defect stream function is linear for most flows of general interest and that this equation is a confluent hypergeometric equation if the flow is in equilibrium and the Clauser outer eddy viscosity model is used. The linear equation pertains for all pressure gradients in the incompressible through supersonic range and for small pressure gradients in the hypersonic range. The modified Crocco temperature-velocity relationship, which accommodates nonunit Prandtl numbers with an empirical recovery factor, has been used as the energy equation for the outer flow.

The law of the wall that was used is a modified form of the van Driest expression for compressible, adiabatic flow, and the law of the wake was developed from numerical data for compressible, adiabatic flow in Ref. 5. However, much more general laws of the wall can be used, and the procedure of Ref. 5 can be used to construct the associated law of the wake.

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